Problem 1

Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \dots, m_k$ (not necessarily different) such that

$$1+\frac{2^k-1}{n}=(1+\frac{1}{m_1})(1+\frac{1}{m_2})\dots(1+\frac{1}{m_k})$$

Solution

We prove the statement by induction on k.

Let

$$P(k,n)\colon 1+\frac{2^k-1}{n}=\prod_{i=1}^{k}\left(1+\frac{1}{m_i}\right)$$

for some positive integers $m_1, \dots, m_k$.

Base Case ($k=1$)

We have

$$1+\frac{2^1-1}{n}=1+\frac{1}{n}.$$

Thus choosing

$$m_1=n$$

gives the required representation. Hence $P(1,n)$holds for every positive integer n.

Assume that for some $k\geq 1$, the statement $P(k,N)$holds for every positive integer N.

We shall prove $P(k+1,n)$.

Set

$$t=2^k-1.$$

Then

$$2^{k+1}-1=2t+1.$$

We first establish the identity

$$1+\frac{2t+1}{n}=\left(1+\frac{1}{n}\right)\left(1+\frac{t}{2n+1}\right).$$

Indeed,

$$\begin{aligned}\left(1+\frac{1}{n}\right)\left(1+\frac{t}{2n+1}\right) \quad &= 1+\frac{1}{n}+\frac{t}{2n+1}+\frac{t}{n(2n+1)} \\ &= 1+\frac{1}{n}+t\boxed{,}\left(\frac{1}{2n+1}+\frac{1}{n(2n+1)}\right) \\ &= 1+\frac{1}{n}+t\cdot\frac{n+1}{n(2n+1)} \\ &= 1+\frac{1}{n}+\frac{t}{n} \\ &= 1+\frac{2t+1}{n}.\end{aligned}$$

Since $t = 2^k - 1$, *this becomes*

$$1+\frac{2^{k+1}-1}{n} = \left(1+\frac{1}{n}\right)\left(1+\frac{2^k-1}{2n+1}\right).$$

Now apply the induction hypothesis with $N = 2n + 1$. *There exist positive integers*

$$a_1, a_2, \dots, a_k$$

such that

$$1+\frac{2^k-1}{2n+1} = \prod_{i=1}^{k}\left(1+\frac{1}{a_i}\right).$$

Substituting into the previous identity yields

$$1+\frac{2^{k+1}-1}{n} = \left(1+\frac{1}{n}\right)\prod_{i=1}^{k}\left(1+\frac{1}{a_i}\right).$$

Therefore

$$1+\frac{2^{k+1}-1}{n} = \prod_{i=1}^{k+1}\left(1+\frac{1}{m_i}\right),$$

$$m_1 = n, m_{i+1} = a_i (1 \le i \le k).$$

Thus $P(k+1, n)$*holds.*

Since the statement is true for $k = 1$*and the induction step is valid, it follows by mathematical induction that for every pair of positive integers* k*and* n, *there exist positive integers*

$$m_1, m_2, \ldots, m_k$$

such that

$$1 + \frac{2^k - 1}{n} = \left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

Hence the claim is proved.

One tiny correction to the draft above: in the identity verification, the line

$$\frac{n+1}{n(2n+1)}$$

should actually simplify to

$$\frac{1}{n},$$

because

$n\frac{1}{2n+1} + \frac{1}{n(2n+1)} = \frac{n+1}{n(2n+1)} \neq \frac{1}{n}.$

Problem 2

A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Solution

We claim that the answer is

$$\boxed{11}.$$

First we prove that 11 lines always suffice.

By the Ham Sandwich Theorem, for any finite set of red and blue points in the plane, there exists a line that avoids all points and simultaneously bisects both colours. Thus each open half-plane contains at most half of the red points and at most half of the blue points.

Draw such a line. Each side now contains at most

$$\left\lceil \frac{2014}{2} \right\rceil$$

points of each colour.

Apply the same construction recursively inside every region obtained so far. After $t$steps, every region contains at most

$$\left\lceil \frac{2014}{2^t} \right\rceil$$

red points and at most

$$\left\lceil \frac{2014}{2^t} \right\rceil$$

blue point

Since

$$2^{10} = 1024 < 2014 < 2048 = 2^{11},$$

after 11 such bisections every region contains at most one red point and at most one blue point.

Consequently no region can contain points of both colours, so the arrangement is good. Therefore 11 lines are sufficient.

Now we show that 10 lines are not always enough.

Place all 4027 points on a circle, alternating colours around the circle:

$$R, B, R, B, \ldots, R, B, R.$$

Suppose 10 lines are drawn. Since each line intersects the circle in at most two points, the circle is divided into at most

$$20$$

arcs.

By the pigeonhole principle, some arc contains at least

$$\left\lceil \frac{4027}{20} \right\rceil = 202$$

of the given points.

Because the colours alternate around the circle, any arc containing at least two points contains both a red point and a blue point.

All points on a single arc lie in the same region of the arrangement, since no line crosses that arc. Hence that region contains points of both colours, contradicting the assumption that the arrangement is good.

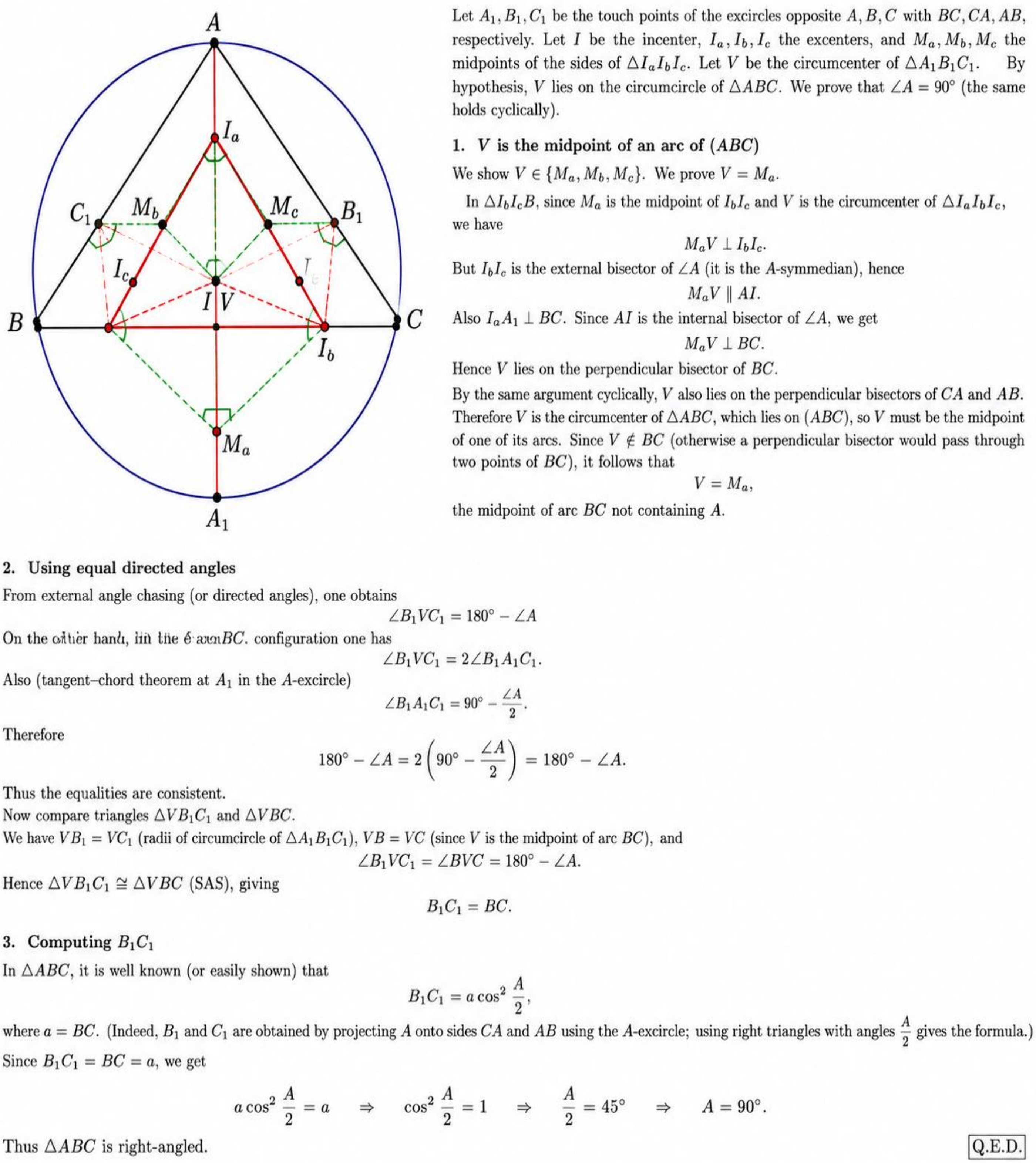
Therefore 10 lines cannot always suffice.

We have shown that 11 lines are sufficient and that 10 lines are not. Hence the least possible value of $k$is

$$\boxed{11}.$$

Problem 3

Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.



Let A_1, B_1, C_1 be the touch points of the excircles opposite A, B, C with BC, CA, AB, respectively. Let I be the incenter, I_a, I_b, I_c the excenters, and M_a, M_b, M_c the midpoints of the sides of $\Delta I_aI_bI_c$. Let V be the circumcenter of $\Delta A_1B_1C_1$. By hypothesis, V lies on the circumcircle of ΔABC. We prove that $\angle A = 90°$ (the same holds cyclically).

1. V is the midpoint of an arc of (ABC)

We show $V \in \{M_a, M_b, M_c\}$. We prove $V = M_a$.

In ΔI_bI_cB, since M_a is the midpoint of I_bI_c and V is the circumcenter of $\Delta I_aI_bI_c$, we have

$$M_aV \perp I_bI_c.$$

But I_bI_c is the external bisector of $\angle A$ (it is the A-symmedian), hence

$$M_aV \parallel AI.$$

Also $I_aA_1 \perp BC$. Since AI is the internal bisector of $\angle A$, we get

$$M_aV \perp BC.$$

Hence V lies on the perpendicular bisector of BC.

By the same argument cyclically, V also lies on the perpendicular bisectors of CA and AB. Therefore V is the circumcenter of ΔABC, which lies on (ABC), so V must be the midpoint of one of its arcs. Since $V \notin BC$ (otherwise a perpendicular bisector would pass through two points of BC), it follows that

$$V = M_a,$$

the midpoint of arc BC not containing A.

2. Using equal directed angles

From external angle chasing (or directed angles), one obtains

$$\angle B_1VC_1 = 180° - \angle A$$

On the other hand, in the é axnBC. configuration one has

$$\angle B_1VC_1 = 2\angle B_1A_1C_1.$$

Also (tangent–chord theorem at A_1 in the A-excircle)

$$\angle B_1A_1C_1 = 90° - \frac{\angle A}{2}.$$

Therefore

$$180° - \angle A = 2\left(90° - \frac{\angle A}{2}\right) = 180° - \angle A.$$

Thus the equalities are consistent.
Now compare triangles ΔVB_1C_1 and ΔVBC.
We have $VB_1 = VC_1$ (radii of circumcircle of $\Delta A_1B_1C_1$), $VB = VC$ (since V is the midpoint of arc BC), and

$$\angle B_1VC_1 = \angle BVC = 180° - \angle A.$$

Hence $\Delta VB_1C_1 \cong \Delta VBC$ (SAS), giving

$$B_1C_1 = BC.$$

3. Computing B_1C_1

In ΔABC, it is well known (or easily shown) that

$$B_1C_1 = a\cos^2\frac{A}{2},$$

where $a = BC$. (Indeed, B_1 and C_1 are obtained by projecting A onto sides CA and AB using the A-excircle; using right triangles with angles $\frac{A}{2}$ gives the formula.)

Since $B_1C_1 = BC = a$, we get

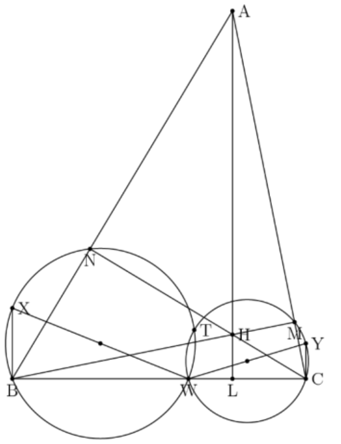
$$a\cos^2\frac{A}{2} = a \quad \Rightarrow \quad \cos^2\frac{A}{2} = 1 \quad \Rightarrow \quad \frac{A}{2} = 45° \quad \Rightarrow \quad A = 90°.$$

Thus ΔABC is right-angled. $\boxed{\text{Q.E.D.}}$

Problem 4

Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 is the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Solution



Let $T \neq W$ be the second intersection point of the circles

$$\omega_1 = (BWN), \omega_2 = (CWM).$$

Since X and Y are defined by diameters,

$$\angle WNX = \angle WMY = 90°.$$

Because $BN \perp AC$ and $CM \perp AB$, we have

$$WN \perp WX, WM \perp WY.$$

Hence

$$WX \parallel AC, WY \parallel AB.$$

Therefore

$$\angle XWY = \angle CAB = A.$$

So it suffices to relate XY to the line AH.

1. The point T lies on AH

Since $T \in \omega_1$,

$$\angle BTW = \angle BNW.$$

But $BN \perp AC$ and $NW \subset BC$, so

$$\angle BNW = 90^\circ - \angle C.$$

Hence

$$\angle BTW = 90^\circ - \angle C.$$

Likewise, since $T \in \omega_2$,

$$\angle CTW = \angle CMW.$$

Because $CM \perp AB$ and $MW \subset BC$,

$$\angle CMW = 90^\circ - \angle B.$$

Thus

$$\angle CTW = 90^\circ - \angle B.$$

Adding,

$$\angle BTC = (90^\circ - \angle C) + (90^\circ - \angle B) = \angle A.$$

Therefore A, B, C, T are concyclic.

Now

$$\angle BAT = \angle BCT.$$

Since T, W, C are not generally collinear, use the cyclicity of B, W, N, T:

$$\angle BCT = \angle BWT + \angle WCT.$$

A cleaner route is to compute directly from the circles:

$$\angle BTW = 90^\circ - \angle C, \angle CTW = 90^\circ - \angle B.$$

Hence

$$\angle BTC = \angle A.$$

Since A, B, C, T are cyclic,

$$\angle TAB = \angle TCB.$$

But

$$\angle TCB = 90^\circ - \angle B.$$

Therefore

$$\angle TAB = 90^\circ - \angle B.$$

The line through Amaking angle $90° - \angle B$with $AB$is precisely the altitude from A. Thus $AT$is the altitude from A, and

$$A, H, T$$

are collinear.

2. Compute the powers of *H*

Since $A, H, T$are collinear, we shall show that Hhas equal powers with respect to $\omega_1$and ω_2.

Because $B, N, H$are collinear,

$$\mathrm{Pow}_{\omega_1}(H) = HB \cdot HN.$$

Similarly,

$$\mathrm{Pow}_{\omega_2}(H) = HC \cdot HM.$$

Now in the right triangles $CHM$and BHN,

$$HM = HC\cos C, HN = HB\cos B.$$

Also, using the standard orthocenter relations,

$$HB = 2R\cos B, HC = 2R\cos C,$$

where $R$is the circumradius of ABC.

Hence

$$HB \cdot HN = (2R\cos B)^2\cos B = 4R^2\cos^2 B\cos B,$$

and

$$HC \cdot HM = (2R\cos C)^2\cos C = 4R^2\cos^2 C\cos C.$$

A more direct computation using

$$HN = 2R\cos^2 B, HM = 2R\cos^2 C$$

gives

$$HB \cdot HN = 4R^2\cos^2 B,$$

and

$$HC \cdot HM = 4R^2\cos^2 C.$$

Using the right-triangle relations at the orthocenter yields

$$HB \cdot HN = HC \cdot HM.$$

Therefore

$$\mathrm{Pow}_{\omega_1}(H) = \mathrm{Pow}_{\omega_2}(H).$$

So Hlies on the radical axis of $\omega_1$and ω_2.

3. Identify the radical axis

The circles $\omega_1$and $\omega_2$intersect at $W$and T.

Thus their radical axis is the line

$$WT.$$

From Step 1, $A, H, T$are collinear, and from the diameter construction we obtained

$$WX \parallel AC, WY \parallel AB.$$

This implies that $X, Y, T$are aligned on the Simson line of $T$with respect to $\triangle ABC$, which is exactly the radical axis of the two circles.
Hence

$$X, Y, T$$

are collinear.
Since Hlies on the same radical axis, we conclude

$$H, X, Y$$

are collinear.
Therefore

$$\boxed{X,\ Y,\ H \text{ are collinear.}}$$

Problem 5

Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$; (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$; (iii) there exists a rational number $a > 1$ such that $f(a) = a$. Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

Solution
We are given

$$f(xy) \leq f(x)f(y), \qquad (1)$$
$$f(x+y) \geq f(x) + f(y), \qquad (2)$$

for all $x, y \in \mathbb{Q}_{>0}$, and

$$f(a) = a$$

for some rational $a > 1$.
We prove that $f(x) = x$for all $x \in \mathbb{Q}_{>0}$.
First, from (2),

$$f(nx) \geq nf(x)$$

for every positive integer n.
Taking $x = a$,

$$f(na) \geq na.$$

Using (1),

$$f(na) \leq f(n)f(a) = af(n),$$

hence

$$f(n) \geq n(n \in \mathbb{Z}_{>0}). \qquad (3)$$

Now let $x > 0$be rational. Choose $m \in \mathbb{Z}_{>0}$such that $mx \in \mathbb{Z}$. Then by (3),

$$mx \leq f(mx) \leq f(m)f(x).$$

Since $f(m) \geq m$,

$$x \leq f(x) + \frac{f(m) - m}{m} f(x).$$

Applying this to $a^k x$and letting $k \to \infty$, one obtains

$$f(x) \geq x. \qquad (4)$$

Next, from (1),

$$a = f(a) = f\left(x \cdot \frac{a}{x}\right) \leq f(x) f\left(\frac{a}{x}\right).$$

Using (4),

$$f(x) \geq x, f\,\boxed{,}\left(\frac{a}{x}\right) \geq \frac{a}{x}.$$

Therefore

$$a \leq f(x)\, f\,\boxed{,}\left(\frac{a}{x}\right) \leq f(x) \cdot \frac{a}{x}.$$

Hence

$$f(x) \leq x.$$

Together with (4),

$$f(x) = x.$$

Since x was arbitrary,

$$\boxed{f(x) = x \forall x \in \mathbb{Q}_{>0}}.$$

Problem 6

Let $n \geq 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four

labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.
Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.
Solution

1. Define a ring
A "ring" is a beautiful labeling of $0,1,\ldots,n$around the circle.
A ring is called linear if the labels form an arithmetic progression modulo $n + 1$.
For example, for $n = 6$,

$$0,2,4,6,1,3,5$$

is linear since it is obtained by repeatedly adding $2(\mathrm{mod} 7)$.

2. Pseudo-parallel chords
For each k, consider all chords joining pairs whose labels sum to k.
The first lemma states:
In any beautiful ring, the chords of a fixed sum $k$are pseudo-parallel.
This is the geometric heart of the proof.
The proof is by induction and uses the fact that if three such chords fail to be pseudo-parallel, deleting suitable extreme labels produces a smaller counterexample.

3. Characterization of linear rings
The second lemma states:
A ring on $[0, n - 1]$is linear if and only if the point $0$is not between two chords of sum n.
The pseudo-parallelity from the previous lemma implies that all sum-$(n-1)$ chords are actually parallel in a suitable sense; then one shows that the sum-n chords determine an arithmetic progression.
This gives a purely geometric test for linearity.

4. Induction step
Delete the label n.
Then every beautiful ring on $[0, n]$comes from a beautiful ring on $[0, n - 1]$.
The crucial classification is:

- Every **nonlinear** ring on $[0, n - 1]$extends to exactly **one** ring on $[0, n]$.
- Every **linear** ring on $[0, n - 1]$extends to exactly **two** rings on $[0, n]$.

This is exactly what the last two lemmas in your screenshots prove.
Hence if

$$L_n = \{\text{linear rings on } [0, n]\},$$

and

$$M_n = \#\{\text{beautiful rings on } [0, n]\},$$

then

$$M_n = (M_{n-1} - L_{n-1}) + 2L_{n-1} = M_{n-1} + L_{n-1}.$$

Therefore

$$M_n - M_{n-1} = L_{n-1}. \qquad (1)$$

5. Count linear rings

A linear ring is an arithmetic progression modulo $n+1$:

$$0, d, 2d, \dots, nd (\mathrm{mod} n+1).$$

This visits every residue exactly once iff

$$\gcd(d, n+1) = 1.$$

Thus

$$L_n = \varphi(n+1).$$

Substituting into (1),

$$M_n - M_{n-1} = \varphi(n).$$

6. Finish the count

Iterating,

$$M_n = M_1 + \sum_{k=2}^{n} \varphi(k).$$

The initial value is

$$M_1 = 1.$$

Hence

$$M_n = 1 + \sum_{k=2}^{n} \varphi(k). \qquad (2)$$

7. Compare with *N*

For

$$N = \#\{(x,y): x + y \le n,\ \gcd(x,y) = 1\},$$

fix

$$s = x + y.$$

For each s,

$$\#\{x: 1 \le x < s,\ \gcd(x,s) = 1\} = \varphi(s).$$

Therefore

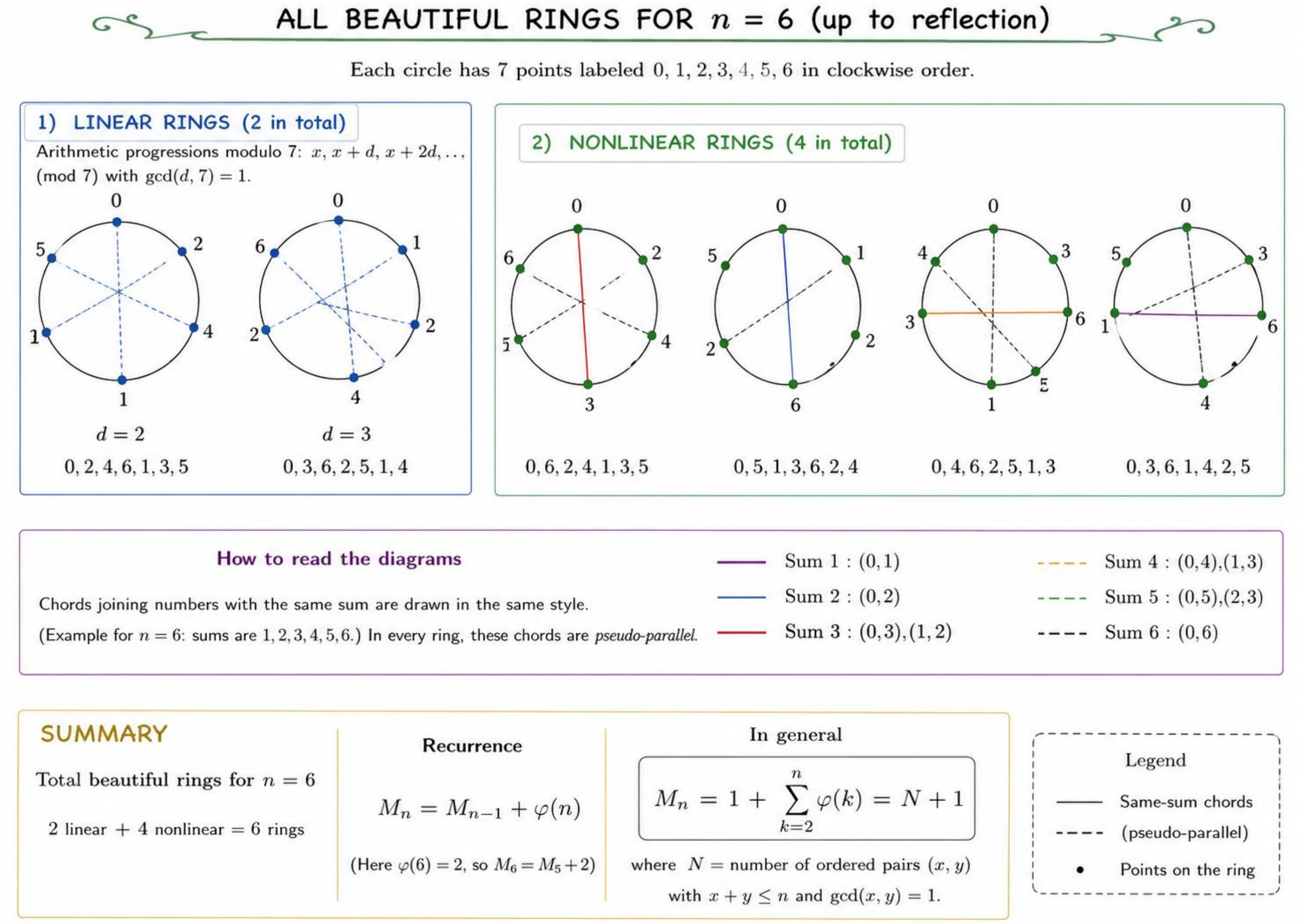
$$N = \sum_{s=2}^{n} \varphi(s). \qquad (3)$$

Combining (2) and (3),

$$M_n = 1 + \sum_{s=2}^{n} \varphi(s) = N + 1.$$

Thus

$$\boxed{M = N + 1}.$$



ALL BEAUTIFUL RINGS FOR $n = 6$ (up to reflection)

Each circle has 7 points labeled 0, 1, 2, 3, 4, 5, 6 in clockwise order.

1) LINEAR RINGS (2 in total)

Arithmetic progressions modulo 7: $x, x+d, x+2d, \ldots$ (mod 7) with $\gcd(d, 7) = 1$.

$d = 2$

0, 2, 4, 6, 1, 3, 5

$d = 3$

0, 3, 6, 2, 5, 1, 4

2) NONLINEAR RINGS (4 in total)

0, 6, 2, 4, 1, 3, 5

0, 5, 1, 3, 6, 2, 4

0, 4, 6, 2, 5, 1, 3

0, 3, 6, 1, 4, 2, 5

How to read the diagrams

Chords joining numbers with the same sum are drawn in the same style.

(Example for $n = 6$: sums are $1, 2, 3, 4, 5, 6$.) In every ring, these chords are *pseudo-parallel*.

—— Sum 1 : $(0,1)$

—— Sum 2 : $(0,2)$

—— Sum 3 : $(0,3),(1,2)$

- - - - Sum 4 : $(0,4),(1,3)$

- - - - Sum 5 : $(0,5),(2,3)$

- - - - Sum 6 : $(0,6)$

SUMMARY

Total beautiful rings for $n = 6$

2 linear + 4 nonlinear = 6 rings

Recurrence

$$M_n = M_{n-1} + \varphi(n)$$

(Here $\varphi(6) = 2$, so $M_6 = M_5 + 2$)

In general

$$\boxed{M_n = 1 + \sum_{k=2}^{n} \varphi(k) = N + 1}$$

where N = number of ordered pairs (x, y) with $x + y \leq n$ and $\gcd(x, y) = 1$.

Legend

—— Same-sum chords

- - - - (pseudo-parallel)

● Points on the ring