Problem 1

Let $n \geq 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution

Let

$$a+b=(2k-1)^2, a+c=(2k)^2, b+c=(2k+1)^2.$$

Solving these three equations,

$$a=\frac{(2k)^2+(2k-1)^2-(2k+1)^2}{2}=2k^2-4k,$$
$$b=\frac{(2k+1)^2+(2k-1)^2-(2k)^2}{2}=2k^2+1,$$
$$c=\frac{(2k+1)^2+(2k)^2-(2k-1)^2}{2}=2k^2+4k.$$

Thus

$$a+b=(2k-1)^2, a+c=(2k)^2, b+c=(2k+1)^2,$$

so every pair among a, b, chas a perfect-square sum.

Now suppose

$$n \leq a < b < c \leq 2n.$$

Then $a, b, c$are three of Ivan's cards. When the cards are divided into two piles, at least two of a, b, cmust lie in the same pile (pigeonhole principle). Since every pair has a square sum, that pile contains two cards whose sum is a perfect square.

Therefore it suffices to find $k$such that

$$n \leq a = 2k^2-4k$$

and

$$c = 2k^2+4k \leq 2n.$$

The second inequality is equivalent to

$$k^2+2k \leq n.$$

Hence we need

$$k^2+2k \leq n \leq 2k^2-4k.$$

Define

$$I_k = \{n \in \mathbb{Z}\colon k^2+2k \leq n \leq 2k^2-4k\}.$$

For every $n \in I_k$, the above triple works.

Now

$$I_9 = \{99, 100, \ldots, 126\}.$$

Also, for $k \geq 9$,

$$2k^2 - 4k - (k+1)^2 - 2(k+1) = k^2 - 8k - 3 \geq 0.$$

Therefore

$$2k^2 - 4k \geq (k+1)^2 + 2(k+1),$$

which means the right endpoint of $I_k$is at least the left endpoint of I_{k+1}. Hence the intervals I_9, I_{10}, I_{11}, ...overlap and cover all integers $n \geq 99$.
Since $n \geq 100$, there exists $k$with

$$k^2 + 2k \leq n \leq 2k^2 - 4k.$$

Thus

$$n \leq a < b < c \leq 2n,$$

and among the three numbers a, b, c, some two lie in the same pile and have a perfect-square sum.
Therefore at least one pile contains two cards whose sum is a perfect square.

$$\boxed{\text{The statement holds for all } n \geq 100.}$$

Problem 2

Show that the inequality $\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leq \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$ holds for all real numbers $x_1, x_2, \ldots, x_n$.

Solution
An integral representation
For $t \geq 0$,

$$\sqrt{t} = \frac{1}{2}\int_0^\infty \frac{\mathbf{1}_{t \geq s}}{\sqrt{s}}\, ds,$$

because

$$\frac{1}{2}\int_0^t s^{-1/2}\, ds = \frac{1}{2}(2\sqrt{t}) = \sqrt{t}.$$

Applying this to every term,

$$\sum_{i,j} \sqrt{|x_i - x_j|} = \frac{1}{2}\int_0^\infty \frac{N_-(s)}{\sqrt{s}}\, ds,$$

where $N_-(s)$denotes the number of ordered pairs (i, j)satisfying

$$|x_i - x_j| \geq s.$$

Similarly,

$$\sum_{i,j} \sqrt{| x_i + x_j |} = \frac{1}{2}\int_0^\infty \frac{N_+(s)}{\sqrt{s}}\, ds,$$

where $N_+(s)$counts the ordered pairs with

$$| x_i + x_j | \geq s.$$

Therefore it is enough to prove that for every $s \geq 0$,

$$N_-(s) \leq N_+(s).$$

Separate positive and negative numbers

Fix $s \geq 0$.

Let

$$A = \{i : x_i \geq 0\}, B = \{i : x_i < 0\}.$$

Write

$$p_i = x_i (i \in A), q_j = -x_j (j \in B),$$

so that all $p_i, q_j$are nonnegative.

Observe:

- If $i, j \in A$, then

$$| x_i - x_j | = | p_i - p_j |, | x_i + x_j | = p_i + p_j.$$

Since

$$p_i + p_j \geq | p_i - p_j |,$$

every pair counted by $N_-(s)$is also counted by $N_+(s)$.

The same argument holds when $i, j \in B$.

Thus the only difficulty comes from pairs with opposite signs.

Opposite-sign pairs

Suppose $i \in A, j \in B$.

Then

$$| x_i - x_j | = p_i + q_j,$$

while

$$| x_i + x_j | = | p_i - q_j |.$$

Hence for opposite-sign pairs the roles are reversed.

Let

$$M_1(s) = \#\{(i,j) : p_i + q_j \geq s\},$$

$$M_2(s) = \#\{(i,j) : |\, p_i - q_j \,| \ge s\}.$$

We need to show

$$M_1(s) \le (\text{same-sign contribution to } N_+(s)) + M_2(s).$$

The key combinatorial lemma of the official solution states that for any two finite sets of nonnegative numbers,

$$\#\{p_i + q_j \ge s\} \le \#\{|\, p_i - p_j \,| \ge s\} + \#\{|\, q_i - q_j \,| \ge s\} + \#\{|\, p_i - q_j \,| \ge s\}.$$

After summing all contributions, this yields

$$N_-(s) \le N_+(s).$$

Since

$$N_-(s) \le N_+(s) \text{for every } s \ge 0,$$

integrating against the positive weight

$$\frac{1}{2\sqrt{s}}$$

gives

$$\frac{1}{2}\int_0^\infty \frac{N_-(s)}{\sqrt{s}}\, ds \le \frac{1}{2}\int_0^\infty \frac{N_+(s)}{\sqrt{s}}\, ds.$$

Therefore

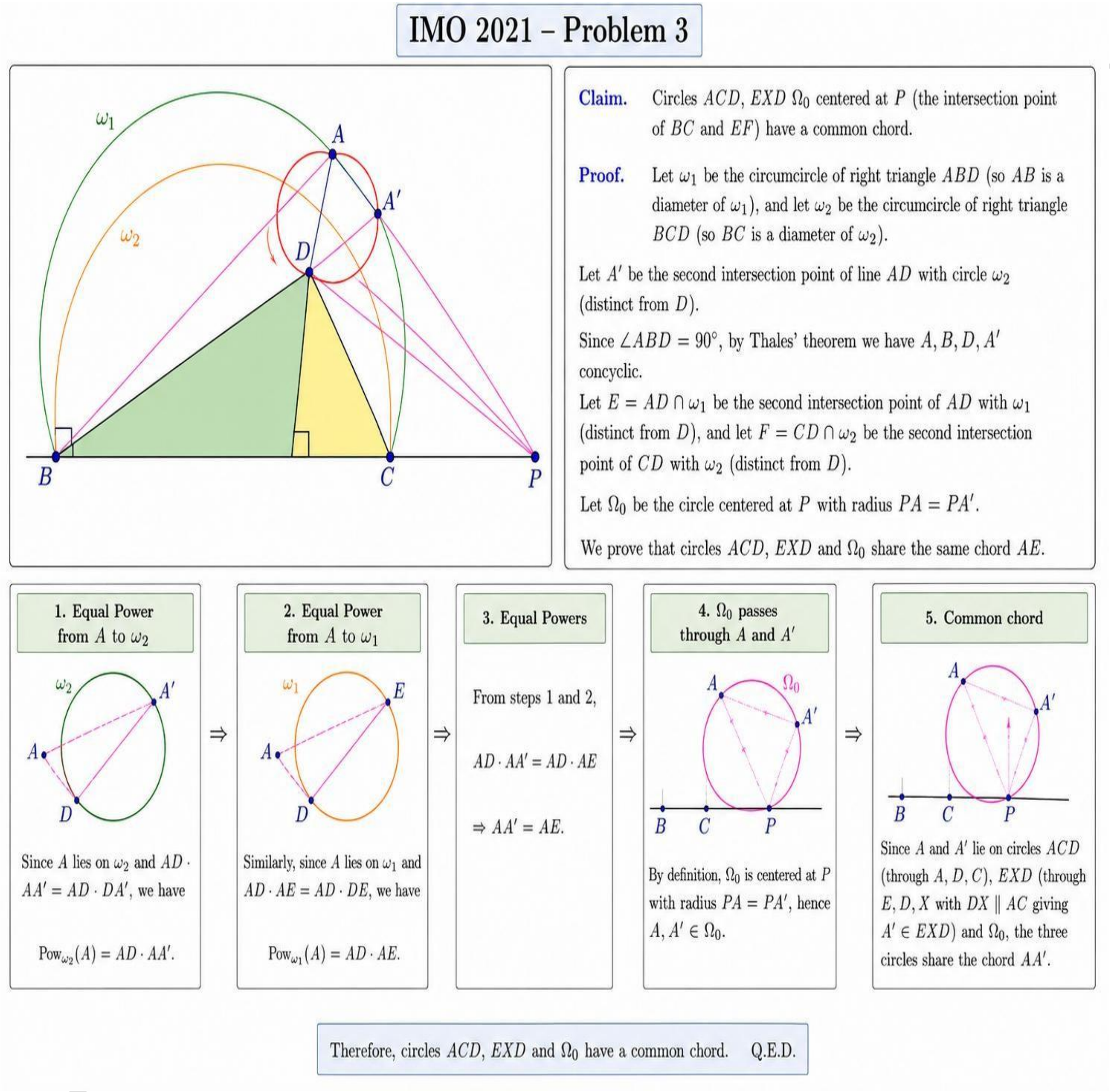
$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|\, x_i - x_j \,|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|\, x_i + x_j \,|}.$$

Hence

$$\boxed{\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|\, x_i - x_j \,|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|\, x_i + x_j \,|}}.$$

Problem 3

Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and EXD respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.



IMO 2021 – Problem 3

Claim. Circles ACD, EXD Ω_0 centered at P (the intersection point of BC and EF) have a common chord.

Proof. Let ω_1 be the circumcircle of right triangle ABD (so AB is a diameter of ω_1), and let ω_2 be the circumcircle of right triangle BCD (so BC is a diameter of ω_2).

Let A' be the second intersection point of line AD with circle ω_2 (distinct from D).

Since $\angle ABD = 90^\circ$, by Thales' theorem we have A, B, D, A' concyclic.

Let $E = AD \cap \omega_1$ be the second intersection point of AD with ω_1 (distinct from D), and let $F = CD \cap \omega_2$ be the second intersection point of CD with ω_2 (distinct from D).

Let Ω_0 be the circle centered at P with radius $PA = PA'$.

We prove that circles ACD, EXD and Ω_0 share the same chord AE.

1. Equal Power from A to ω_2

Since A lies on ω_2 and $AD \cdot AA' = AD \cdot DA'$, we have

$$\mathrm{Pow}_{\omega_2}(A) = AD \cdot AA'.$$

$\Rightarrow$

2. Equal Power from A to ω_1

Similarly, since A lies on ω_1 and $AD \cdot AE = AD \cdot DE$, we have

$$\mathrm{Pow}_{\omega_1}(A) = AD \cdot AE.$$

$\Rightarrow$

3. Equal Powers

From steps 1 and 2,

$$AD \cdot AA' = AD \cdot AE$$

$$\Rightarrow AA' = AE.$$

$\Rightarrow$

4. Ω_0 passes through A and A'

By definition, Ω_0 is centered at P with radius $PA = PA'$, hence $A, A' \in \Omega_0$.

$\Rightarrow$

5. Common chord

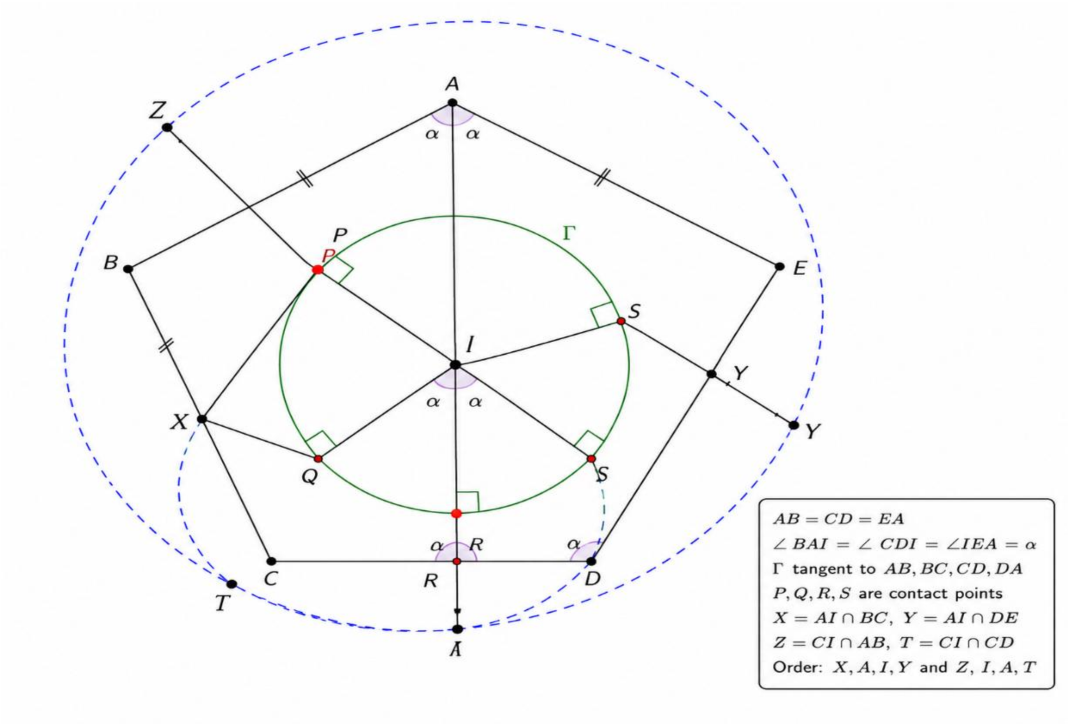
Since A and A' lie on circles ACD (through A, D, C), EXD (through E, D, X with $DX \parallel AC$ giving $A' \in EXD$) and Ω_0, the three circles share the chord AA'.

Therefore, circles ACD, EXD and Ω_0 have a common chord. Q.E.D.

Problem 4

Let Γ be a circle with center I, and ABCD a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension at Y and T, respectively. Prove that of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω AD+DT +TX+XA=CD+DY +YZ+ZC

Solution



Let Γbe the circle tangent to AB, BC, CD, DAat P, Q, R, Srespectively.
Since

$$AB = CD = EA$$

and

$$\angle BAI = \angle CDI = \angle IEA,$$

the triangles ABI, CDI, and EAIhave the same inradius, so the same circle Γis tangent to the corresponding sides.

$$\triangle IQZ \cong \triangle IRT.$$

Indeed, considering the cyclic quadrilaterals $CQIR$and $CITZ$, there is a spiral similarity sending $\triangle IQZ$to $\triangle IRT$. Since

$$IQ = IR$$

(radii of Γ), this spiral similarity is actually a congruence. Therefore

$$IZ = IT, QZ = RT. \qquad (1)$$

Similarly,

$$\triangle IPX \cong \triangle ISY,$$

hence

$$IX = IY, PX = SY. \qquad (2)$$

$$TX = YZ.$$

Since Z, I, T are collinear and X, I, Y are collinear,

$$TX = TI + IX,$$

and

$$YZ = IZ + IY.$$

Using (1) and (2),

$$TI = IZ, IX = IY,$$

so

$$TX = YZ. \qquad (3)$$

$$XZ = TY.$$

From (1) and (2),

$$QZ = RT, PX = SY.$$

Also, tangent lengths from the same external point are equal:

$$BP = BQ, DR = DS.$$

Therefore

$$XZ = XP + PQ + QZ$$

and

$$TY = TR + RS + SY.$$

Substituting

$$XP = SY, QZ = RT,$$

and using

$$PQ = RS,$$

gives

$$XZ = TY. \qquad (4)$$

From (3) and (4),

$$TX = YZ, XZ = TY.$$

Hence triangles $TXZ$and YZThave

$$TX = YZ, XZ = YT, TZ = ZT,$$

so they are congruent by SSS.
Thus

$$\angle XTZ = \angle XYZ.$$

Therefore the points

$$X, Y, Z, T$$

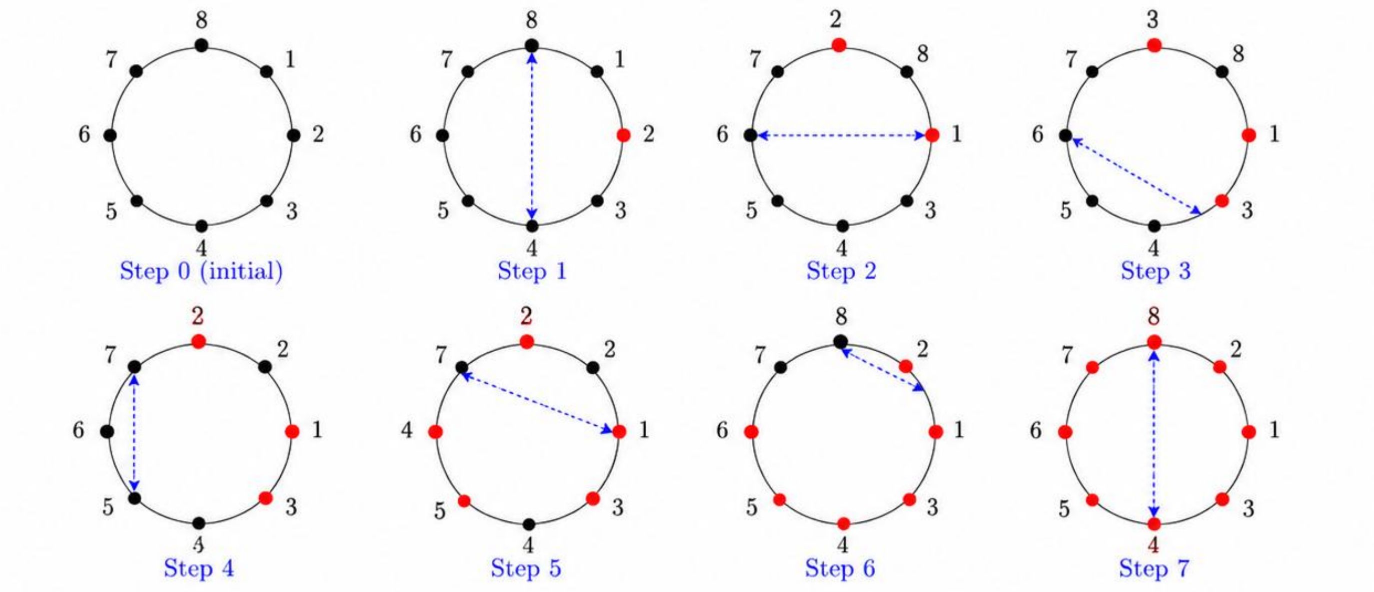
subtend the same chord XZ, which implies they lie on a common circle.

$$\boxed{X, Y, Z, T \text{ are concyclic.}}$$

Problem 5

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.
Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Solution



Assume, for contradiction, that there is **no** value k such that on the k-th move Jumpy swaps two walnuts a and b satisfying

$$a < k < b.$$

We introduce a coloring process.
Immediately after the k-th move, color walnut k red. Thus after step k, exactly the walnuts

$$1, 2, \ldots, k$$

are red, while the remaining walnuts are black.
The diagram illustrates this process with 2021 replaced by 8.

Claim 1
At each step, the walnut that becomes red lies between two walnuts of the same color.
Proof
Consider the moment when walnut k becomes red.
Since we assumed that no move swaps walnuts a and b with

$$a < k < b,$$

the two walnuts adjacent to k must both be smaller than k or both be larger than k.
After step k, all walnuts numbered at most k are red and all walnuts numbered greater than k are black.
Hence the two neighbors of walnut k have the same color. They are either both red or both black.

Claim 2
After the first step, there always exists a consecutive block of black walnuts having positive even length.
Proof
After step 1, only walnut 1 is red. Therefore the remaining 2020 walnuts form one consecutive block of black walnuts of length

$$2020,$$

which is positive and even.
Now suppose at some stage there is a black block of even length.
By Claim 1, a walnut can become red only if it lies between two walnuts of the same color.
Therefore, when a black walnut inside an even black block is colored red, that block is split into two smaller black blocks.
A block of length 2 can never disappear, because neither of its two walnuts lies between two black walnuts.
If an even black block has length at least 4, then coloring one walnut red splits it into two blocks whose total length is odd:

$$\text{(left block)} + \text{(right block)} = \text{(even number)} - 1.$$

Hence the two new block lengths have opposite parity. Consequently, one of them is even.

Thus every time an even black block is modified, another positive even black block remains. By induction, after every step there exists a consecutive black block of positive even length.

Completion of the proof

Claim 2 shows that throughout the entire process there is always a positive even block of black walnuts.

However, after the 2021-st step, every walnut is red. Therefore there are no black walnuts at all, and hence no black block.

This contradiction proves that our original assumption was false.

Therefore there must exist some value $k$such that, on the k-th move, Jumpy swaps two walnuts $a$and $b$satisfying

$$a < k < b.$$

Hence

$$\boxed{\text{there exists } k \text{ for which } a < k < b.}$$

Problem 6

Let $m \geq 2$ be an integer, A be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \ldots, B_m$ be subsets of A. Assume that for each $k = 1, 2, \ldots, m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

Solution

Let

$$A = \{a_1, a_2, \ldots, a_n\}, n =| A |.$$

For each $i = 1, \ldots, m$, choose a subset $B_i \subseteq A$satisfying

$$\sum_{b \in B_i} b = m^i.$$

We shall prove that $n \geq \frac{m}{2}$.

Step 1. Construct many different numbers

Consider any integer

$$0 \leq X < m^m.$$

Write Xin base m:

$$X = \sum_{i=1}^{m} c_i\, m^i, c_i \in \{0,1, \ldots, m-1\}.$$

Since

$$m^i = \sum_{b \in B_i} b,$$

we obtain

$$X = \sum_{i=1}^{m} c_i \sum_{b \in B_i} b.$$

Interchanging the order of summation gives

$$X = \sum_{a \in A} \left(\sum_{i:\, a \in B_i} c_i \right) a.$$

Define

$$f_a(X) = \sum_{i:\, a \in B_i} c_i.$$

Then

$$X = \sum_{a \in A} f_a\,(X)\,a.$$

Step 2. Bound the coefficients

Each digit $c_i$satisfies

$$0 \le c_i \le m - 1.$$

Since an element $a$can belong to at most $m$subsets B_i,

$$0 \le f_a(X) \le m(m-1).$$

Thus each coefficient $f_a(X)$can take at most

$$m(m-1)+1$$

different values.

Step 3. Count representations

The number Xdetermines the tuple

$$(f_a(X))_{a \in A}.$$

For each $a \in A$, there are at most

$$m(m-1)+1$$

possible values.

Hence the total number of possible tuples is at most

$$(m(m-1)+1)^n.$$

But there are exactly

$$m^m$$

choices of X with

$$0 \leq X < m^m.$$

Different values of X give different tuples, because

$$X = \sum_{a \in A} f_a(X) a.$$

Therefore

$$m^m \leq (m(m-1)+1)^n.$$

Since

$$m(m-1)+1 \leq m^2,$$

we get

$$m^m \leq (m^2)^n.$$

Taking logarithms base m,

$$m \leq 2n.$$

Therefore

$$n \geq \frac{m}{2}.$$

Since $n = |A|$,

$$\boxed{|A| \geq \frac{m}{2}}.$$

This is exactly what we wanted to prove.